Tschebotareff's theorem on the product of non-homogeneous linear forms

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§ 1. Let

$$L_r(x) + c_r = \sum_{s=1}^{n} a_{rs} x_s + c_r, \quad (r = 1, 2, \ldots, n) \tag{1}$$

be n non-homogeneous linear forms, where the $a's$ and $c's$ are real and where, without loss of generality, the determinant $\| a_{rs} \| = 1$.

An unproved hypothesis due to MINKOWSKI asserts that there exists a lattice point (x), i. e. a set of integer values for the $x's$, such that

$$\prod_{r=1}^{n} | L_r(x) + c_r | \leq \frac{1}{2^n}. \tag{2}$$

In a lecture(1) at the Oslo International Mathematical Congress in 1936, I recalled that MINKOWSKI had proved the result for $n = 2$ and that various simple proofs were known; and that for $n = 3$ a very lengthy and complicated proof[1]) was given by REMAK(2). I also said I knew of no general result when $n \geq 4$, not even the existence of a number M, independent of the coefficients a and c, such that (2), with M in place of $1/2^n$, has always a lattice point solution.

In a letter to me in October 1937, Prof. SIEGEL gave an account of his new method for reducing a set of homogeneous linear forms. He applied this to prove the existence of an M without, however, giving any estimate for it. I gave an account of his method at my seminar, and it suggested to DAVENPORT(4) a method which led to an estimate for $M(n)$, the lower bound of admissible numbers M, namely

[1]) Last year, DAVENPORT(3) gave an exceedingly simple proof of REMAK'S result.

$$M(n) \leq \left(n\, 2^{n-1}\, \Gamma\left(1+\frac{1}{2}\,n\right)(n!)^{\frac{n-1}{2}} / \Gamma\left(\frac{1}{2}\right)^{n}\right)^{n}. \tag{3}$$

Shortly afterwards, Prof. TSCHEBOTAREFF drew my attention to a paper [(5)] of his published in 1934 in Russian in a rather inaccessible Russian journal, which had escaped the notice of the reviewers, in which he proved the important result

$$M(n) \leq 2^{-\frac{1}{2}n}. \tag{4}$$

His method is so simple that it is surprising that it escaped notice for so long a time. I notice now that his proof can be slightly modified, and by using a recent result of mine, I prove [2])

$$M(n) \leq 2^{-\frac{1}{2}n} / [1+(\sqrt{2}-1)^{n}]. \tag{5}$$

§ 2. Let m^n be the lower bound of the left hand side of (2) for lattice points (x). Then for arbitrary $\varepsilon > 0$, a lattice point (x) exists such that

$$\prod_{r=1}^{n} |\, L_r(x) + c_r \,| < (m+\varepsilon)^n,$$

and on changing the origin, we may suppose (x) is the origin (0). Hence, we have, say

$$m^n \leq c^n = |\, c_1\, c_2 \ldots c_n \,| < (m+\varepsilon)^n. \tag{6}$$

On replacing $L_r(x) + c_r$ by $\frac{c}{c_r}\,(L_r(x)+c_r)$, which does not alter the product of the linear forms (1), or the determinant of the a's, we may suppose that the c's are all equal to c. Hence we are given that for all lattice points (x)

$$\prod_{r=1}^{n} |\, L_r(x) + c \,| \geq m^n > (c-\varepsilon)^n, \tag{7}$$

(where the $L_r(x)$ now depend on ε), and replacing (x) by $(-x)$ and multiplying, we have

$$\prod_{r=1}^{n} |\, L_r^2(x) - c^2 \,| \geq (c-\varepsilon)^{2n}. \tag{8}$$

The use of this product is TSCHEBOTAREFF's idea, but now I proceed rather differently.

Apply the inequality of the arithmetic and geometric means to the square of (8). Then

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[2]) This result was also found independently by DAVENPORT a few days after I found it. His method involves ideas not very dissimilar from mine. Dr. DAVENPORT also informs me that he has since found a slightly better result.

$$n(c-\varepsilon)^4 \leq \sum_{r=1}^{n} [L_r^2(x) - c^2]^2,$$

$$2c^2 \sum_{r=1}^{n} L_r^2(x) \leq \sum_{r=1}^{n} L_r^4(x) + nc^4 - n(c-\varepsilon)^4,$$

Hence, if $(x) \neq (0)$,

$$2c^2 \leq \frac{\sum_{r=1}^{n} L_r^4(x)}{\sum_{r=1}^{n} L_r^2(x)} + \frac{\varepsilon'}{\sum_{r=1}^{n} L_r^2(x)}$$

say, and so

$$2c^2 \leq \max_{r=1,\ldots,n} L_r^2(x) + \varepsilon' / \sum_{r=1}^{n} L_r^2(x). \tag{9}$$

Suppose now that λ is a number independent of ε and is such that the inequalities

$$|L_r(x)| \leq \lambda, \quad (r = 1, 2, \ldots, n \tag{10}$$

have a lattice-point solution $(y) \neq (0)$. We may then assume for at least one of the L's that $|L(y)| \geq \frac{1}{2}\lambda$. This is clear on taking, if need be, instead of (y) the lattice point (py), where p is a suitably chosen integer.

Let the (x) in (9) be (y) and take the limit of both sides as $\varepsilon \to 0$. Then

$$2m^2 \leq \lambda^2 \tag{11}$$

If $\lambda = 1$, it is well known that (10) has a solution and so (11) gives TSCHEBOTAREFF'S result $2m^2 \leq 1$.

§ 3. To improve this result, I use the following theorem which I published [5] a few years ago.

Let $L_r(x)$ and c_r be as in (1) and let μ_r, ν_r $(r = 1, 2, \ldots, n)$ be two sets of positive numbers such that

$$\prod_{r=1}^{n} \mu_r + \prod_{r=1}^{n} \nu_r \geq 1. \tag{12}$$

Then there exists a solution of at least one of the three sets of inequalities

(A) $\quad |L_r(x)| \leq \mu_r, \quad (x) \neq (0), \quad (r = 1, 2, \ldots, n)$

(B) $\quad |L_r(x)| \leq \nu_r, \quad (x) \neq (0), \quad (r = 1, 2, \ldots, n)$

(C) $\quad |L_r(x) + c_r| \leq \frac{1}{2}(\mu_r + \nu_r) \quad (r = 1, 2, \ldots, n)$

Take all the μ's equal to ϱ, say, and all the ν's equal to $\sqrt[n]{1-\varrho^n}$, so that (12) is satisfied if $0<\varrho<1$.

Hence we deduce from (A), (B) and (11)

$$2m^2 \leq \varrho^2, \quad 2m^2 \leq \sqrt[n]{(1-\varrho^n)^2},$$

and from (C)

$$m \leq \frac{1}{2}(\varrho + \sqrt[n]{1-\varrho^n}).$$

Take ϱ now such that

$$\frac{\varrho}{\sqrt{2}} = \frac{1}{2}(\varrho + \sqrt[n]{1-\varrho^n}),$$

or

$$\varrho^n = \frac{1}{1+(\sqrt{2}-1)^n}.$$

Since $\varrho^n > \frac{1}{2}$, $\varrho^2 > \sqrt[n]{(1-\varrho^n)^2}$,

$$m^n \leq \frac{1}{2^{\frac{1}{2}n}(1+(\sqrt{2}-1)^n)},$$

and so (5) is proved.

(1) L. J. MORDELL, "MINKOWSKI's theorems and hypotheses on linear forms". Comptes Rendus du Congrès International des Mathématiciens (1) (1936), 236—238.

(2) R. REMAK, «Verallgemeinerung eines Minkowski'schen Satzes». I. Math. Z. 17 (1923) 1—34, II. Math. Z. (18) 137—200.

(3) H. DAVENPORT, "A simple proof of REMAK's theorem on the product of three linear forms". Journal of the London Mathematical Society, 14 (1939), 47—51.

(4) H. DAVENPORT, "Note on a result of SIEGEL". Acta Arithmetica, 2 (1937), 262—265.

(5) N. G. TSCHEBOTAREFF, "Note on algebra and the theory of numbers". Scientific notes of Kazan University, 94 (1934), 14—16. (In Russian.) For a German translation of the paper, see this volume, pg. 27.

(6) L. J. MORDELL, "An arithmetical theorem on linear forms". Acta Arithmetica, 2 (1937), 173—176.