Remarks on Structures and Group Relations.

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§ 1.

A structure has been defined as a system of elements A, B, C, ... with two unique operations $A \cup B$ and $A \cap B$ named union and cross-cut respectively[1]). These two operations satisfy the axions

$$(1)\quad \begin{array}{ll} A \cup B = B \cup A & A \cap B = B \cap A \\ A \cup A = A & A \cap A = A \\ A \cup (B \cup C) = (A \cup B) \cup C & A \cap (B \cap C) = (A \cap B) \cap C \\ A \cup (A \cap B) = A & A \cap (A \cup B) = A. \end{array}$$

The two operations are dual. We also write $A \supset B$ if any one of the two equivalent relations $A \cup B = A$, $A \cap B = B$ holds.

A structure is said to be a DEDEKIND structure if the following condition is satisfied: for any three elements A, B, C with $C \supset A$

$$(2)\qquad C \cap (A \cup B) = A \cup (C \cap B).$$

It was already shown by DEDEKIND[2]) that in such a structure any three elements A, B, C generate a special structure with at most 28 elements. These elements may be obtained by permutations of A, B, C from the following.

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[1]) In the following we presuppose the results of the first chapter of the paper: OYSTEIN ORE: On the Foundation of Abstract Algebra. I. Annals of Math. vol. 36 (1935) pp. 406—439.

[2]) DEDEKIND: Über die von drei Moduln erzeugte Dualgruppe. Math. Ann. 53 (1900). Werke, vol. 2, pp. 371—403.

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\begin{array}{lrcl}
 & & A & \\
 & (A \cup [B \cap C]) \cap (B \cup C) & = & (A \cap [B \cup C]) \cup (B \cap C) \\
 & A \cup B \cup C & , & A \cap B \cap C \\
(3) & (A \cup B) \cap (B \cup C) \cap (A \cup C) & , & (A \cap B) \cup (B \cap C) \cup (A \cap C) \\
 & (A \cup B) \cap (A \cup C) & , & (A \cap B) \cup (A \cap C) \\
 & A \cup (B \cap C) & , & A \cap (B \cup C) \\
 & A \cup B & , & A \cap B.
\end{array}
$$

§ 2.

Now we suppose that A, B, C are elements of an arbitrary structure $\sum$ and we wish to determine the necessary and sufficient conditions which have to be satisfied in order that these three elements generate a DEDEKIND substructure of $\sum$. In order to determine these conditions we form all elements (3). When all possible unions and cross-cuts of these elements are taken, the same relations must be satisfied as in a DEDEKIND structure, and these are then the conditions we are seeking. However, several of these conditions are trivial in the sense that they follow already from the structure axioms (1), for instance, $(A \cup B) \cup (B \cup C) = A \cup B \cup C$. Others are essential in the sense that they require the DEDEKIND relation in some form. By going through the rather tedious process of trying all possibilities and obtaining all conditions one obtains after weeding out some conditions which are obviously consequences of the others:

Theorem 1. The necessary and sufficient conditions that three elements A, B, C in an arbitrary structure shall generate a DEDEKIND substructure is that the following relations together with all relations obtained by permutations and their duals shall hold:

$$
\begin{array}{ll}
\alpha) & (A \cup [B \cap C]) \cap (B \cup C) = (A \cap [B \cup C]) \cup (B \cap C) \\
(4)\ \beta) & (A \cup B) \cap (A \cup C) = A \cup (B \cap [A \cup C]) \\
\gamma) & (A \cup B) \cap (A \cup C) \cap (B \cup C) = (A \cap [B \cup C]) \cup (B \cap [A \cup C]).
\end{array}
$$

Counting the duals this gives altogether five different types of relations to be satisfied. We shall not here go into the question of their independence.

These remarks also suggest a number of conditions weaker than the DEDEKIND axiom which one can impose on a structure. Let us mention only

$$
A \cup (B \cap [A \cup C]) = A \cup (C \cap [A \cup B])
$$
$$
(A \cap [B \cup C]) \cup (B \cap [A \cup C]) = (B \cap [A \cup C]) \cup (C \cap [A \cup C]).
$$

§ 3.

We shall now apply the preceding remarks to groups and determine certain conditions under which the subgroups A, B, C of a given group generate a DEDEKIND structure.

Let us recall that two subgroups A and B are permutable if for any a and b in A and B respectively we have $a \cdot b = b_1 \cdot a_1$. If A and B are permutable the DEDEKIND relation holds:

$$\text{if} \quad C \supset A, C \cap (A \cup B) = A \cup (C \cap B).$$

We shall also need the following lemma: If A and B are permutable and $\bar{A} \supset A$ then A and $B \cap \bar{A}$ are permutable[3]).

We shall now prove:

Theorem 2. Let A, B and C be subgroups of a given group such that each of them is permutable with the two others and with their cross-cut. Then the structure generated by A, B and C is a DEDEKIND structure.

Proof: We shall have to prove that all the relations (4) hold. The relations α are seen to hold because A is permutable with $B \cap C$. The relations β hold because A and B are permutable. Since $B \cap (A \cup C)$ is permutable with A by the lemma just mentioned, we can write the right-hand side of γ in the form $(B \cup C) \cap (A \cup [B \cap [A \cup C])$ and γ follows from β. Furthermore α is self-dual. The duals of β and γ follow again from the permutability of A and $B \cap C$.

As an application of theorem 2 we mention:

Theorem 3. In a group two normal subgroups A and B and an arbitrary third subgroup C generate a DEDEKIND structure.

Proof: Since a normal subgroup is permutable with any other subgroup, the three groups A, B and C are permutable with each other and for the same reason A is permutable with $B \cap C$ and B with $A \cap C$. Finally C is permutable with $A \cap B$ since this last group is normal.

§ 4.

When the relation $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ holds for any three elements in a structure the structure is said to be

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[3]) See OYSTEIN ORE: Structures and Group Theory I, Duke Math. Journal, vol. 3 (1937), p. 159.

distributive. The distributive structure generated by three elements consists of the elements obtainable by permutations from the following.

$$
\begin{array}{lc}
 & A \\
 & (A \cup B) \cap (A \cup C) \cap (B \cup C) = (A \cap B) \cup (A \cap C) \cup (B \cap C) \\
(5) & A \cup B \quad , \quad A \cap B \\
 & (A \cap B) \cup (A \cap C) \ , \ (A \cup B) \cap (A \cup C).
\end{array}
$$

Now considering the necessary and sufficient conditions for three elements in an arbitrary structure to form a distributive structure, one finds by forming unions and cross-cuts of the elements (5):

Theorem 4. The necessary and sufficient condition that three elements A, B and C in an arbitrary structure generate a distributive structure is that the following relations, together with all relations obtained by permutions, shall hold

$$
(6) \quad \begin{array}{l}
(A \cup B) \cap (A \cup C) \cap (B \cup C) = (A \cap B) \cup (A \cap C) \cup (B \cap C) \\
(A \cup B) \cap (A \cup C) = A \cup (B \cap C) \, , \, (A \cap B) \cup (A \cap C) = A \cap (B \cup C).
\end{array}
$$

In a Dedekind structure the conditions become simpler:

Theorem 5. The necessary and sufficient condition that three elements in a Dedekind structure shall generate a distributive structure is that any single one of the relations (6) shall hold.

This means for instance that if A and B are normal subgroups and C is an arbitrary group such that $C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$, then A, B and C generate a distributive structure and all the other relations (6) must hold.